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In a recent paper Kinber, Smith, Velauthapillai, and Wiehagen introduced a new notion of "parallel learning." They call a set S of functions (m, n)-*learnable* if there is a learning machine which for any n-tuple of pairwise distinct functions from S learns at least m functions correctly from examples of their behavior after seeing some finite amount of input. One of the basic open questions in this area is the "inclusion problem," i.e., the question for which m, n, h, k, every (m, n)-learnable class is also (h, k)-learnable. In this paper we develop a general approach to solve this problem. The idea is to associate with each m, n, h, k in a uniform way a finite 2-player game such that the first player has a winning strategy in this game iff every (m, n)-learnable class is (h, k)-learnable. In this way we take the recursion theoretic disguise off the problem and isolate its combinatorial core. We also explicitly characterize the "strength" of each particular noninclusion by the complexity of an oracle which is needed to overcome it. It turns out that there are exactly three different types of noninclusions.

1. INTRODUCTION

In a recent paper Kinber, Smith, Velauthapillai, and Wiehagen [11, 12] introduced a new notion of "parallel learning" to model the learning of a collection of concepts all chosen from a single set. More precisely, they call a set S of functions (m, n)-*learnable* if there is a learning machine which for any n-tuple of pairwise distinct functions from S learns at least m functions correctly from examples of their behavior after seeing some finite amount of input. One of the basic questions in this area is the "inclusion problem," i.e., the question for which m, n, h, k, every (m, n)-learnable class is also (h, k)-learnable. This question turned out to be difficult and in [11, 12] it could be solved only for a few instances.

In this paper we propose a general approach for attacking this and similar problems. From the analysis of particular cases one gets the impression that the core of the problem is purely combinatorical and can be separated from the recursion theoretic part which appears to be more or less invariant. However, formalizing and proving this conjecture in full generality is a nontrivial task. In our case, the combinatorial part is game theoretical: We associate with each m, n, h, k in a uniform way a finite 2-player game $G(m, n; h, k)$. In this game, the moves of Player 1 model an (m, n)-machine which diagonalizes an (h, k)-machine and the moves of Player 2 model an (h, k)-machine which simulates an (m, n)-machine. The point is that we have modeled these processes in a finitary way. In particular, there is an algorithm to determine which player has a winning strategy in $G(m, n; h, k)$. In this way we have taken the recursion theoretic disguise off the problem and have isolated its combinatorial core.

This works out nicely for the *popperian version* of parallel learning (where all guesses have to be total functions) and we get a complete characterization of the corresponding inclusion problem: Noninclusions correspond to winning strategies of Player 1 and inclusions correspond to winning strategies of Player 2. Moreover, the game $G(m, n; h, k)$ can be analyzed explicitly for some interesting values of the parameters. In this way we also obtain several families of inclusions and noninclusions in explicit form.

For the general (not necessarily popperian) case a slight modification of the game gives us a strong sufficient condition for noninclusions which allows an explicit solution of the "equality problem," i.e., the question for which m, n, h, k, (m, n)-learnable and (h, k)-learnable coincide.

In the popperian case we are also able to explicitly characterize the "strength" of each particular noninclusion by the complexity of an oracle which is needed to overcome it. This means, for each choice of parameters m, n, h, k, we characterize the oracles A such that any popperian (m, n)-machine can be simulated by a popperian (h, k)-machine which has access to A. We present a general framework for this type of questions which is based on the distinction between on-line and off-line strategies. Roughly, if a noninclusion is witnessed by an off-line strategy of Player 1, then an oracle for the halting problem is needed to overcome it. If a noninclusion is witnessed by an on-line strategy but not by an off-line strategy, then strictly weaker oracles are sufficient, namely any oracle that computes a complete and consistent extension of Peano arithmetic.

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1.1. *Notation and Definitions*

The set of all natural numbers is denoted by ω. The set of all finite sequences of natural numbers is ω^*. $\sigma * \tau$ is the concatenation of σ and τ, for $\sigma, \tau \in \omega^*$. Sometimes we simply write 131 for $1 * 3 * 1$, etc. We write $\sigma \preccurlyeq \tau$ if σ is an initial segment of τ. The set ω^* can be identified with an infinite tree whose nodes are ordered by $\preccurlyeq$. The root of this tree is the empty sequence λ. The functions $f\colon \omega \to \omega$ can be identified with infinite branches of ω^*. The initial segment of f of length t is denoted by $f \upharpoonright t$; i.e., $f \upharpoonright t$ is the finite function with domain $\{0, ..., t-1\}$ which agrees with f on its domain. The recursion theoretic notation is standard and follows the books [20, 24]. Let REC be the set of all total recursive functions.

We recall the definitions of some well-known inference criteria; see [21] for further background. An inductive inference machine (IIM) M is a total recursive function with domain ω^* and range $\omega \cup \{?\}$. *M finitely infers $f \in$ REC* if there exists $t \in \omega$ such that $M(f \upharpoonright s) = ?$, for all $s < t$, and $M(f \upharpoonright t) = e$, where e is an index of f, i.e. $\varphi_e = f$. In this case we also write $M(f) = e$. We say that M diverges on input f if $M(f \upharpoonright t) = ?$, for all $t \in \omega$. M finitely infers $S \subseteq$ REC iff M finitely infers all $f \in S$. Intuitively, after reading a certain finite initial segment of $f \in S$, M knows an index of f. FIN $= \{S \subseteq \text{REC} : (\exists M)\ [M \text{ finitely infers } S]\}$. This notion was introduced by Gold [7].

An IIM M is called *popperian* if every number in range(M) is an index of a total recursive function; see [2, Definition 2.16]. PFIN is the class of all $S \subseteq$ REC which can be finitely inferred by a popperian IIM.

Below we consider a slight generalization of IIMs which take as input initial segments of n functions in parallel and output n-tuples of programs.

1.2. *Basic Definitions and Facts for Parallel Learning*

In this section we give the formal definition of parallel learning and review the known results on the inclusion problem obtained by Kinber *et al.* [11].

DEFINITION 1.1 [11]. Let S be a set of recursive functions, $1 \leqslant m \leqslant n$. S is *finitely* (m, n)*-learnable* iff there is an inductive inference machine M that takes as input any pairwise distinct functions $f_1, ..., f_n \in S$ and computes an n-tuple of indices $e_1, ..., e_n$ such that at least m of them are correct, i.e., satisfy $f_i = \varphi_{e_i}$. Formally,

$$(\forall \text{ distinct } f_1, ..., f_n \in S)\ (\exists t, e_1, ..., e_n)\ (\forall s < t),\ [M(f_1 \upharpoonright s, ..., f_n \upharpoonright s) = ?,\quad M(f_1 \upharpoonright t, ..., f_n \upharpoonright t) = \langle e_1, ..., e_n\rangle,\quad |\{i\colon \varphi_{e_i} = f_i\}| \geqslant m].$$

Let (m, n) FIN be the class of all S that are finitely (m, n)-learnable. Furthermore, let (m, n) PFIN be the class of all S that are finitely (m, n)-learnable via some popperian IIM, M.

Remark. Note that the definition requires the IIM to converge on all n functions. If this requirement is relaxed such that some components may output "?" forever *and* simultaneous convergence is not required, then one gets a strictly weaker notion.

The following fact summarizes the "easy inclusions."

Fact 1.2 [11, Theorem 12]. The following inclusions hold for FIN:

- $(m+1, n+1)$ FIN $\subseteq$ (m, n) FIN $\subseteq$ $(m, n+1)$ FIN

and

- (m, n) FIN $\cap$ (h, k) FIN $\subseteq (m+h, n+k)$ FIN.

The same inclusions also hold for PFIN:

- $(m+1, n+1)$ PFIN $\subseteq (m, n)$ PFIN $\subseteq (m, n+1)$ PFIN

and

- (m, n) PFIN $\cap$ (h, k) PFIN $\subseteq (m+h, n+k)$ PFIN.

Trivially, (n, n) FIN = FIN, (n, n) PFIN = PFIN.

The next fact provides an important family of noninclusions. It generalizes the well-known result that classes which contain an accumulation point are not finitely learnable.

Fact 1.3 [11, Lemma 7]. If $n - m > k - h$ then the following noninclusions hold:

- (m, n) FIN $\nsubseteq (h, k)$ FIN and
- (m, n) PFIN $\nsubseteq (h, k)$ PFIN.

From the previous results we get an easy explicit solution of the inclusion problem "(m, n) FIN $\subseteq (h, k)$ FIN" for $n \geqslant k$. In the following sections we deal with the case when $n < k$, which turns out to be much more intricate.

COROLLARY 1.4. *Let $n \geqslant k$. Then* (m,n) FIN $\subseteq (h,k)$ FIN $\Leftrightarrow n - m \leqslant k - h \Leftrightarrow (m, n)$ PFIN $\subseteq (h, k)$ PFIN.

Proof. If $n \geqslant k$ and $n - m \leqslant k - h$ we get (m, n) FIN $\subseteq$ $(m-1, n-1)$ FIN $\subseteq (m-(n-k), k)$ FIN $\subseteq (h, k)$ FIN by Fact 1.2. Note that the last inclusion is trivial since $h \leqslant m - (n-k)$. (The other direction is the contrapositive of Fact 1.3.) ∎

2. A GAME THEORETICAL CHARACTERIZATION OF THE INCLUSION PROBLEM FOR PARALLEL LEARNING

In this section we provide game-theoretical characterizations of the inclusion problems for PFIN and FIN. The idea of using finite games to solve recursion theoretic questions was previously employed in the investigation of the lattice of r.e. sets by Degtev [4] and Lachlan [17].

2.1. *A Characterization of the Inclusion Problem for PFIN*

We begin with a general definition of finite games. Then we define the specific versions that characterize the inclusion problem of parallel learning. We will return to general finite games in Section 4.

DEFINITION 2.1. A finite two person game $\mathscr{G}$ is a 5-tuple (G_1, G_2, W, s_0, t_0) such that $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are finite directed acyclic graphs, $W \subseteq V_1 \times V_2$, and $(s_0, t_0) \in (V_1 \times V_2) - W$.

We say that node v is *adjacent* to w in a directed graph G iff there is a directed path in G from v to w (the path may be empty, i.e., v is adjacent to v; we say that w is *properly* adjacent to v if w is adjacent to v and $v \neq w$).

The game (G_1, G_2, W, s_0, t_0) is played in rounds as follows. There are two players: Anke and Boris. At the beginning Anke has a marker at node $s_0 \in V_1$ and Boris has a marker at node $t_0 \in V_2$. A *position* is just an element of $V_1 \times V_2$. So the starting position is (s_0, t_0). In each round both players move their markers to some adjacent node. Boris moves first. All previous moves are known to both players. The position after Boris' move must belong to W. Anke is not allowed to perform empty moves. The first player who is unable to move according to these rules loses the game. By the restriction on the moves of Anke, it is clear that the game ends after at most $|V_1|$ rounds. Since the game is finite, one of the players has a winning strategy.

Intuitively, each time when Boris is to move he wants to "equalize" by moving to a position in W. Anke tries to obtain a position from which Boris cannot equalize. Since she must perform proper moves, she has to realize her goal in at most $|V_1|$ steps.

We will now describe for the parameters m, n, h, k with $1 \leqslant m \leqslant n \leqslant k$ and $1 \leqslant h \leqslant k$ a finite game $G(m, n; h, k)$ for which we prove that Boris has a winning strategy iff (m, n) PFIN $\subseteq (h, k)$ PFIN. This characterizes the inclusion problem for PFIN. Since the game is finite, one can effectively decide which player has a winning strategy. Thus the inclusion problem for PFIN is decidable.

For the sake of readability we formulate our game not quite according to Definition 2.1 but in a more intuitive way.

DEFINITION 2.2. Let $1 \leqslant m \leqslant n \leqslant k$ and $1 \leqslant h \leqslant k$. In the game $G(m, n; h, k)$ there are two players, Anke and Boris. Each of them is equipped with several movable markers. For every n-element set $D \subseteq \{1, ..., k\}$ and every $j \in D$, Anke has a marker $\mu_{D,j}$. Boris has k markers $v_1, ..., v_k$.

The markers are moved on the "infinite board" ω^*. At the beginning each $\mu_{D,j}$ is placed on node j (here j is considered as a sequence of length 1 in ω^*, i.e., as the root of the subtree $j * \omega^*$) and each v_j is placed on node λ. In each move a player is allowed to shift her (his) markers downwards in the tree to adjacent nodes. Boris moves first.

Anke is only allowed moves of the following type ("node splittings"). She selects a node σ which contains at least two of her markers and distributes all of her markers from σ onto the successor nodes $\sigma * 1, ..., \sigma * a$, for some $a \geqslant 2$, such that each of these nodes receives at least one marker.

Boris chooses for each of his markers v_j an adjacent node $\sigma_j \succcurlyeq j$, containing at least one marker of Anke, and moves v_j to node σ_j. Furthermore, at any time Boris may move one of his markers from any node σ to node 0 (and stay there forever); this is only for technical reasons, we need it to model a silly move of Boris.

Note that after each move of Boris any two markers either belong to incomparable nodes or they belong to the same node.

In order to determine the winner of the game we need the following notion: The markers are in an *A-configuration via* $L \subseteq \omega^*$ iff

- Every node in L contains a marker of Anke and for each $j = 1, ..., k$ there is at most one node $\sigma \succcurlyeq j$ in L.
- For every D, at least m of Anke's markers $\mu_{D,1}, ..., \mu_{D,n}$ are on nodes in L.
- Less than h of Boris' markers v_j are on nodes in L.

The other configurations of the game are called *B-configurations.* Boris wins the game iff after each of his moves the markers are in a B-configuration.

Intuitively, Boris is trying to establish with each of his moves a B-configuration, while Anke tries to eventually establish an A-configuration which cannot be transformed in a B-configuration by any of Boris' moves.

Note that Anke has exactly $p = n\binom{k}{n}$ markers, which are initially distributed on k nodes. Every move of Anke increases the number of nodes which contain at least one of her markers; so after Anke has moved j times at least $k + j$ different nodes contain one of her markers. Thus Anke cannot make more than $p - k$ moves and the game ends after at most $1 + p - k$ rounds. Therefore we do not really need an infinite board, the finite tree $\bigcup_{s \leqslant p} \{0, ..., p\}^s$ would be enough.

Now it is easy to see that we can reformulate $G(m, n; h, k)$ as a finite game according to Definition 2.1; W corresponds to the set of all B-configurations, etc.

THEOREM 2.3. *Let* $k \geqslant n$. *Then* (m, n) PFIN $\subseteq (h, k)$ PFIN *iff Boris has a winning strategy in* $G(m, n; h, k)$.

Proof. $(\Rightarrow)$ We show the contrapositive. Assume that Boris has no winning strategy. Since the game is finite, Anke has a winning strategy. Furthermore, we may assume that if Anke plays according to her winning strategy, then after each of her moves she reaches an A-configuration. We show that this winning strategy is the basic building block to construct a class $S \in (m, n)$ PFIN $-$ (h, k) PFIN by diagonalization.

Let $\{M_i\}_{i \in \omega}$ be a recursive listing of all inductive inference machines. We define S inductively and add for every i a set of k functions which is not (h, k) PFIN-inferred by M_i. This diagonalizes every (h, k) PFIN algorithm. It should be noted that S is defined nonuniformly. This idea is due to Kinber *et al.* [11, 12] who used it in their proofs to $(b-1, b)$ FIN $\not\subseteq$ BC and $(1, 2)$ FIN $\not\subseteq (2, 3)$ FIN.

To ensure that $S \in (m, n)$ PFIN we construct a uniformly recursive family of total recursive functions

$$\{F_{i, e, D, j}\colon i, e \in \omega \wedge D \subseteq \{1, ..., k\} \wedge |D| = n \wedge j \in D\}$$

and a (nonuniform) family $\{f_{i, e, j}\colon i, e \in \omega \wedge 1 \leqslant j \leqslant k\}$ of total recursive functions with the following properties:

(I) $f_{i, e, j}(0) = \langle i, e, j \rangle$ and $f_{i, e, j}(x) = 0$ for almost all $x > 0$.

(II) For all $D \subseteq \{1, ..., k\}$, $|D| = n$, there are m distinct indices $j_1, ..., j_m \in D$ such that $f_{i, e, j_1} = F_{i, e, D, j_1}$, $f_{i, e, j_2} = F_{i, e, D, j_2}$, ..., $f_{i, e, j_m} = F_{i, e, D, j_m}$.

(III) M_i does not (h, k) PFIN-infer $f_{i, e, 1}, ..., f_{i, e, k}$.

S is the ascending union of finite sets S_i: Let $S_0 = \varnothing$. Suppose we have already defined the finite set S_i. Then, by (I), there is a constant e_i such that for all $f \in S_i$, $f(x) = 0$ for $x \geqslant e_i$. Let $S_{i+1} = S_i \cup \{f_{i, e_i, 1}, ..., f_{i, e_i, k}\}$.

$S \in (m, n)$ PFIN. Consider any n pairwise different functions $g_1, ..., g_n \in S$. The inference algorithm first reads $g_1(0), ..., g_n(0)$ which gives the corresponding values $\langle i_1, e_{i_1}, j_1 \rangle, ..., \langle i_n, e_{i_n}, j_n \rangle$. Let e be the maximum of these e_i's. Then the algorithm reads the initial segments of length e of each function; w.l.o.g., let $g_1, ..., g_u$ be the functions with maximal first component $i = i_1, ..., i_u$. For the remaining functions g_j with $j > u$ we have $i_j < i$. Thus, by the definition of e_i, $g_j = (g_j \upharpoonright e_i) * 0^\omega$ for $u < j \leqslant n$. Hence we can compute the indices of these functions which gives us $n - u$ correct components.

Since $j_1, ..., j_u$ are pairwise different there is $E \subseteq \{1, ..., k\}$, $|E| = n - u$ such that $D = \{j_1, ..., j_u\} \cup E$ is an n-element set. Then we output in the first u components the indices of $F_{i, e, D, j_1}, ..., F_{i, e, D, j_u}$. By (II), at least $m - (n - u)$ of them are correct. So we get a total of m correct components as required.

$S \notin (h, k)$ PFIN. In stage i of the construction of S, functions $f_{i, e_i, 1}, ..., f_{i, e_i, k}$ are added to S which are not (h, k) PFIN-inferred by M_i. So there is not IIM M such that $S \in (h, k)$ PFIN via M.

It remains to construct the functions $F_{i, e, D, j}$ such that (I), (II), and (III) are satisfied. Let i, e be fixed. We implement a diagonalization of M_i guided by the winning strategy of Anke. $F_{i, e, D, j}$ is constructed as follows. We define in stages a play of the game $G(m, n; h, k)$ such that the positions of Anke's markers correspond to the functions $F_{i, e, D, j}$ and the positions of Boris' markers correspond to the functions guessed by M_i.

More formally, for every node σ of the board and each stage s we define a value $\tau_s(\sigma)$. This value is either undefined or it is a finite function defined on an initial segment. If $\tau_s(\sigma)$ is defined then $\tau_{s+1}(\sigma)$ is also defined and satisfies $\tau_s(\sigma) \preccurlyeq \tau_{s+1}(\sigma)$. Let $\mu_{D, j, s}$ denote the position of marker $\mu_{D, j}$ at stage s. Then we define

$$F_{i, e, D, j} = \lim_s \tau_s(\mu_{D, j, s}) = \tau(\lim_s \mu_{D, j, s}). \qquad (*)$$

The position $\nu_{j, s}$ of marker ν_j at stage s is defined from the functions ψ_j which denote the functions guessed by M_i:

$$\psi_j(x) = \begin{cases} \varphi_{e_j}(x) & \text{if } \; M_i(\langle i, e, 1 \rangle * 0^\omega, ..., \langle i, e, k \rangle * 0^\omega) \\ & \quad = \langle e_1, ..., e_k \rangle \\ \uparrow & \text{otherwise } \; (M_i(\langle i, e, 1 \rangle * 0^\omega, ..., \\ & \quad \langle i, e, k \rangle * 0^\omega) \text{ diverges}). \end{cases}$$

W.l.o.g. we assume that $\psi_{j, s}(x)$ is undefined if $M_i(\langle i, e, 1 \rangle * 0^t, ..., \langle i, e, k \rangle * 0^t) = ?$ for all $t < s$, or if $x \geqslant s$. In every stage s the diagonalization procedure does the following:

- Check whether Boris has moved.
- If so, check whether the game is in a B-configuration.
- If both conditions are satisfied, implement Anke's next move.
- Extend τ in order to make all functions total.

Anke's markers are at each stage on the *leaves* of the tree spanned by the positions of all markers. The other nodes are called interior nodes. If σ becomes an *interior node* in stage s then $\tau_{s'}(\sigma) = \tau_s(\sigma)$ for all $s' \geqslant s$. Only the τ values of the leaves may change. Furthermore, if $\sigma \preccurlyeq \sigma'$, $s \leqslant s'$ and $\tau_s(\sigma)$, $\tau_{s'}(\sigma')$ are defined, then $\tau_s(\sigma) \preccurlyeq \tau_{s'}(\sigma')$. Now we present the algorithm in detail:

(1) *Initialize the algorithm.* Let $\tau_0(\lambda) = \lambda$ and $\tau_0(j) = \langle i, e, j \rangle$ for $j = 1, ..., k$. Furthermore, put Anke's markers $\mu_{D, j}$ on node j and Boris' markers on the root λ. Let $s = 0$.

(2) *Reconstruct the positions of Boris' markers.* Note that $\mathrm{dom}(\psi_{j, s}) \subseteq \{0, ..., s\}$ and $\mathrm{dom}(\tau_s(\sigma)) \supseteq \{0, ..., s\}$ for every leaf σ. For every marker ν_j define its position $\nu_{j, s}$ as follows:

$$\nu_{j, s} = \begin{cases} \sigma & \text{if there is } \sigma \in \mathrm{dom}(\tau_s) \text{ such that } j \preccurlyeq \sigma \text{ and} \\ & \quad \sigma \text{ is the shortest string with } \psi_{j, s} \preccurlyeq \tau_s(\sigma); \\ 0 & \text{otherwise (no } \tau_s(\sigma) \text{ extends } \psi_{j, s}). \end{cases}$$

We shall see later that if the first case occurs then σ is uniquely determined. The 0 in the second case stands for markers no longer to be considered in the game; it means that for each leaf σ there is $x \leqslant s$ such that $\psi_{j,s}(x)\downarrow \neq \tau_s(\sigma)(x)$ and thus $\psi_j \neq F_{i,e,D,j}$ for all j and D. In particular if a marker v_j is once moved onto 0, it remains there forever; i.e., $v_{j,t}=0$ for all $t \geqslant s$.

(3) *Check whether Boris has completed his move.* A move of Boris is complete only if all of his markers are in the leaves and if the game is in a B-configuration. If this has not already been achieved, go to step (5); otherwise continue at step (4).

(4) *Implement Anke's move according to the winning strategy.* Since the game is in a B-configuration, Anke selects, according to the winning strategy, a leaf σ and distributes the markers of node σ onto the nodes $\sigma * 1, ..., \sigma * a$ $(a \geqslant 2)$.

(5) *Extend τ on the leaves*:

$$\tau_{s+1}(\sigma) = \begin{cases} \tau_s(\sigma) & \text{if } \sigma \text{ is an interior node;} \\ \tau_s(\sigma) * 0 & \text{if } \sigma \text{ is an old leaf (i.e., } \tau_s(\sigma)\downarrow\text{);} \\ \tau_s(\eta) * b & \text{if } \sigma = \eta * b \text{ is a new leaf from step (4)} \\ & \quad \text{(i.e., } \tau_s(\sigma)\uparrow\text{);} \\ \uparrow & \text{otherwise.} \end{cases}$$

Let $s = s+1$ and go to step (2).

Note that $\mu_{D,j,s}$ is always placed on a leaf. By induction on s and the update rule for τ_s, it follows that $|\tau_s(\sigma)| \geqslant s$ for all leaves σ. Therefore the function $F_{i,e,D,j} = \lim_s \tau_s(\mu_{D,j,s})$ are total. If σ, η are incomparable nodes and $\tau_s(\sigma)$, $\tau_s(\eta)$ are both defined, then they are also incomparable. Further, if α is the largest common prefix of σ and η, then $\tau_s(\alpha)$ is the largest common prefix of $\tau_s(\sigma)$, $\tau_s(\eta)$. This ensures that $v_{j,s}$ is uniquely defined.

Anke moves only finitely often. After her last move she reaches an A-configuration. Choose a set of nodes L witnessing the A-configuration. Boris cannot reach a B-configuration (otherwise Anke would need at least one further move to win the game). Therefore Boris will never complete his last move. On the other hand, there are only finitely many stages where he moves his markers. Let s be sufficiently large such that after stage s no marker is moved and consider the final configuration in stage s.

(a) If one of Boris' markers v_j is neither on a leaf nor on node 0, then the corresponding ψ_j is not total; M_i is not a PFIN-machine. In this case we let $f_{i,e,j} = \lim_t \tau_t(\eta)$ if $\eta \in L$ and $\eta \succcurlyeq j$. If there is no $\eta \in L$ with $\eta \succcurlyeq j$, we let $f_{i,e,j} = \langle i, e, j\rangle\, 0^\omega$. Then it is easy to see that (I), (II), and (III) are satisfied.

(b) Assume that in the final configuration every v_j is on a leaf or on node 0. We let $f_{i,e,j} = \lim_t \tau_t(\eta)$ if $\eta \in L$ and $\eta \succcurlyeq j$. If there is no $\eta \in L$ with $\eta \succcurlyeq j$, then we choose $f_{i,e,j} \succcurlyeq \langle i, e, j\rangle * 0^s$ such that $f_{i,e,j}$ is almost always zero and different from ψ_j.

Obviously, conditions (I) and (II) are satisfied. Suppose for a contradiction that M_i (h,k) PFIN-infers $f_{i,e,1}, ..., f_{i,e,k}$. Let $t \geqslant 1$ be minimal such that $M_i(f_{i,e,1}\upharpoonright t, ..., f_{i,e,k}\upharpoonright t)$ is defined, say, equal to $\langle e'_1, ..., e'_k\rangle$. By construction, $f_{i,e,j}\upharpoonright t = \langle i, e, j\rangle * 0^{t-1}$ and therefore $(e'_1, ..., e'_k) = (e_1, ..., e_k)$. Thus, at least h of the equations $f_{i,e,j} = \psi_j$ must hold. If there is no $\eta \in L$ with $\eta \succcurlyeq j$ then, by definition, $f_{i,e,j} \neq \psi_j$. Thus there are at least h nodes $\eta \in L$ such that $\psi_j = \lim_t \tau_t(\eta)$. However, since the τ values of incomparable nodes are incomparable, it follows that $\psi_j = \lim_t \tau_t(\eta)$ holds only if the final position of v_j is η. Thus in the final configuration at least h of Boris' markers are on nodes in L. This contradicts the hypothesis that the final configuration is an A-configuration via L. Hence (III) holds.

$(\Leftarrow)$ Assume that Boris has a winning strategy in $G(m,n;h,k)$ and $S \in (m,n)$ PFIN via M. We describe an (h,k) PFIN-machine N which infers S.

Given k pairwise different functions $f_1, ..., f_k$, N simulates $M(f_{i_1}, ..., f_{i_n})$ for every n-element set $D = \{i_1 < \cdots < i_n\} \subseteq \{1, ..., k\}$. N waits until M converges for each such D, say with output $e_{D,i_1}, ..., e_{D,i_n}$. By hypothesis, all of these programs compute total functions. Let $F_{D,j}$ denote the function computed by $e_{D,j}$.

Then N outputs programs which compute the functions $g_1, ..., g_k$ defined as follows: We consider the $F_{D,j}$'s, translate them into configurations of the game, move the markers according to the winning strategy, and translate the positions $v_{i,s}$ of v_i back into g_i: $g_{i,s} = \tau_s(v_{i,s})$.

(1) *Initialization.* Place the markers $\mu_{D,j}$ and v_j on node j. Let $\tau_0(j) = \lambda$, $s=0$, $x=0$, and go to step (2).

(2) *Check whether Anke has moved.* Select a leaf η such that $x = |\tau_s(\eta)|$ is minimal among the lengths $|\tau_s(\sigma)|$ of all leaves σ. For every marker $\mu_{D,j}$ placed on η calculate $F_{D,j}(x)$. Since the guesses $F_{D,j}$ are always total functions, these calculations terminate. Let $y_1, ..., y_a$ be the values. If $a > 1$ then we discovered a move of Anke and go to step (4). Otherwise, Anke did not move, and we go to step (3).

(3) *Adjust τ while waiting for Anke's move.* Since Anke did not move, the game remains in a B-configuration and the only activity is to update τ:

$$\tau_{s+1}(\sigma) = \begin{cases} \tau_s(\sigma) * y_1 & \text{if } \sigma = \eta; \\ \tau_s(\sigma) & \text{otherwise } (\sigma \neq \eta). \end{cases}$$

Let $s = s+1$ and go to step (2).

(4) *Implement Anke's move.* The computations of the $F_{D,j}(x)$ with $\mu_{D,j}$ placed on the leaf η give several different

values $y_1, ..., y_a$. Now τ is adjusted on the new leaves $\eta * b$ $(b = 1, ..., a)$ as follows:

$$\tau_{s+1}(\sigma) = \begin{cases} \tau_s(\eta) * y_b & \text{if } \sigma = \eta * b; \\ \tau_s(\sigma) & \text{otherwise } (\sigma \neq \eta * b \text{ for all } b). \end{cases}$$

All markers $\mu_{D,j}$ with $F_{D,j}(x) \succcurlyeq \tau_s(\eta) * y_b$ move from η to $\eta * b$. Go to step (5).

(5) *Implement Boris' move.* If Boris has no marker on η then he does not move. Otherwise some marker v_i remained on η while all markers of Anke moved to some leaf. Then Boris moves this marker according to his winning strategy from η to a new leaf $\eta * b$. Now the game is again in a B-configuration. Let $s = s + 1$ and go to step (2).

Anke makes only finitely many moves. Therefore the game ends in a B-configuration and for all leaves σ of this final configuration, $\tau_s(\sigma)$ is extended infinitely often. Since every v_i eventually moves onto such a leaf, all $g_i = \lim_s \tau_s(v_{i,s})$ are total. Thus N is a PFIN-machine.

Now suppose that $f_1, ..., f_k \in S$. Let $L = \{\sigma : (\exists j)[f_j = \lim_s \tau_s(\sigma)]\}$. Since the f_j's are total functions, the nodes $\sigma \in L$ must be leaves of the final configuration. Since for every n-element set D, m of functions $F_{D,j}$ coincide with f_j, m of the markers $\mu_{D,j}$ are placed on nodes in L. Thus, h of the markers v_j must be placed on nodes in L, since otherwise the final configuration would be an A-configuration via L. Therefore $g_j = f_j$ for these $v_j \in L$, so N infers at least h of the $f_1, ..., f_k$. Thus, $S \in (h, k)$ PFIN. ∎

2.2. *Noninclusions for FIN*

In this section we define a slight modification of $G(m, n; h, k)$ which is used to give a sufficient condition for the noninclusion (m, n) FIN $\not\subseteq (h, k)$ FIN.

DEFINITION 2.4. The game $G'(m, n; h, k)$ is a variant of the game $G(m, n; h, k)$. The players receive the same markers: Anke has for every n-element set $D \subseteq \{1, ..., k\}$ and each $j \in D$ a marker $\mu_{D,j}$; Boris has the markers $v_1, ..., v_k$. Anke's markers $\mu_{D,j}$ are initially placed on node j and Boris' markers, on the root λ. As in the game G, the markers move on the tree ω^* from nodes σ to adjacent nodes $\eta \succcurlyeq \sigma$. From now on the words leaf, interior node, and successor refer to the subtree generated by the current positions of Anke's markers.

The definition of an A-configuration via a set L is the same as in the game G, but the implicit requirement that L consists of leaves must be made explicit since Anke's markers may remain on interior nodes:

- Every node in L is a leaf (and therefore contains a marker of Anke).
- For each $j = 1, ..., k$ there is at most one node $\sigma \succcurlyeq j$ in L.
- For every D, at least m of Anke's markers $\mu_{D,1}, ..., \mu_{D,n}$ are on nodes in L.
- Fewer than h of Boris' markers v_j are on nodes in L.

The rules to move the markers are less restrictive:

- Anke moves her markers from nodes σ to any adjacent node $\eta \succcurlyeq \sigma$.
- Boris moves his markers from σ to $\eta \succcurlyeq \sigma$ or to 0, where η is inside the subtree generated by Anke's markers and markers on 0 do never leave this node.
- After Anke's move the game is in an A-configuration, after Boris' move it is in a B-configuration.

The players move alternately and Boris has the first move. Boris wins the game if he always moves into a B-configuration; otherwise the game comes to an end in an A-configuration and Anke wins the game.

THEOREM 2.5. *If Anke has a recursive winning strategy for the game $G'(m, n; h, k)$ then (m, n) FIN $\not\subseteq (h, k)$ FIN.*

Proof sketch. The diagonalization works as in Theorem 2.3. In general it is the same except that the $F_{i,e,D,j}$ may be partial, the conditions (I), (II), and (III) are the same, also their verification after the algorithm to implement the winning strategy is similar. Again $F_{i,e,D,j} = \lim_s \tau_s(\mu_{D,j,s})$ and $\psi_{j,s} \preccurlyeq \tau_s(v_{j,s})$ if $v_{j,s} \neq 0$. The algorithm has to be partially adapted:

(1) *Initialize the algorithm.* Place the markers $\mu_{D,j}$ and v_j on node j. Let $\tau_0(j) = \lambda$, $s = 0$, $x = 0$, and go to step (2).

(2) *Reconstruct the positions of Boris' markers.* Let $v_{j,s}$ be the shortest string $\sigma \in \mathrm{dom}(\tau_s)$ such that $\sigma \succcurlyeq j$ and $\psi_{j,s} \preccurlyeq \tau_s(\sigma) * 0^s$; if such a string does not exist let $v_{j,s} = 0$.

(3) *Check whether Boris has completed his move.* If the game is in a B-configuration then Boris completed his move and the algorithm continues at step (5); otherwise go the step (4).

(4) *Extend τ on the leaves while waiting for Boris' move*:

$$\tau_{s+1}(\sigma) = \begin{cases} \tau_s(\sigma) * 0 & \text{if } \sigma \text{ is a leaf;} \\ \tau_s(\sigma) & \text{if } \sigma \text{ is an interior node;} \\ \uparrow & \text{otherwise.} \end{cases}$$

Let $s = s + 1$ and go to step (2).

(5) *Implement Anke's move according to the winning strategy.* Since the game is in a B-configuration, Anke moves the markers according to her winning strategy from nodes σ onto nodes $\sigma' \in \sigma * \{1, 2, ...\}^*$. The game is in an A-configuration again. Go to step (6).

(6) *Update τ for stage $s+1$ after Boris' and Anke's moves.* Let "old tree" refer to the tree generated by Anke's marker positions before step (5) and let "new tree" refer to the tree of the marker positions after step (5). Every node σ in the new tree can be split into an old part η which is the longest initial segment of σ belonging to the old tree and a new part η' defined by the equation $\sigma=\eta * \eta'$. If σ already belonged to the old tree then $\eta'=\lambda$; otherwise $\eta' \in \{1, 2, ...\}^+$. The update rule for τ is

$$\tau_{s+1}(\sigma)=\begin{cases}\tau_s(\eta)*0^s*\eta' & \text{if } \sigma \text{ is in the new tree, but is} \\ & \quad \text{not an interior node of the old tree;} \\ \tau_s(\eta) & \text{if } \sigma \text{ is an interior node of the old tree;} \\ \uparrow & \text{otherwise } (\sigma \text{ is not in the new tree}).\end{cases}$$

Let $s=s+1$ and go to step (2).

Since Anke follows in step (5) a recursive winning strategy, this strategy can be coded into the programs of the $F_{i,e,D,j}$'s. Further, by the winning strategy, she moves only finitely often. After Anke's last move, Boris has only finitely many possibilities to shift his markers but he will not reach a B-configuration. So the game ends in a final A-configuration at some stage s witnessed via some set L of leaves. Now functions $f_{i,e,j}$ are defined via L as in Theorem 2.3 and the further verification of the local step is analogous. Note that the $f_{i,e,j}$'s are total since $\lim_t \tau_t(\eta)$ is total if η is a leaf at stage s. Those ψ_j, which belong to markers v_j remaining on an interior node at stage s, are not total. ∎

A further modification is the game G'' which is a version in between G and G'. The only difference between G'' and G is that Boris—as in the game G'—is not required to move all markers onto leaves while Anke's moves have to fulfil the same requirements as in the game G. Also the definition of A-configuration and B-configuration is the same as in game G. A small modification of the proof of Theorem 2.3 gives that (m, n) PFIN $\subseteq (h, k)$ FIN iff Boris has a winning strategy for the game $G''(m, n; h, k)$.

We do not know whether the game $G'(m, n; h, k)$ characterizes the inclusion problem for FIN; also it is not a finite game. However, the inclusion problem for FIN is decidable by reducing it to an infinite game on a finite graph [18]. The details are worked out in [22]. Note that by now we cannot guarantee that there are any nontrivial inclusions for FIN besides those that follow from Fact 1.2. If this were indeed the case then one would have an easy explicit description of the inclusion structure and no games would be needed. In contrast, we do not expect that there is an explicit description of the inclusion problem for PFIN (e.g., there is the nontrivial inclusion $(4, 5)$ PFIN $\subseteq (5, 6)$ PFIN of Theorem 3.2 below).

Open Problem. Are there any inclusions for FIN besides those that follow from Fact 1.2?

There are certain partial results on the way to this conjecture. Proposition 3.5 shows that $(n, n+1)$ FIN $\not\subseteq$ $(n+1, n+2)$ FIN. Furthermore, Corollary 3.10 establishes the conjecture for $m=1$: $(1, n)$ FIN $\subseteq (h, k)$ FIN iff $k \geqslant hn$. For $m=2$ we can show as a first result that $(2, n)$ FIN $\subseteq$ $(3, k)$ FIN iff $k \geqslant 2n-1$. But already the questions whether

- $(2, n)$ FIN $\subseteq (5, k)$ FIN $\Leftrightarrow k \geqslant 3n-1$ and
- $(3, n)$ FIN $\subseteq (4, k)$ FIN $\Leftrightarrow k \geqslant 2n-2$

are open.

3. EXPLICIT RESULTS ON THE INCLUSION PROBLEM FOR PARALLEL LEARNING AND POPPERIAN PARALLEL LEARNING

The next results are applications of the game theoretic characterizations of the inclusion relation.

3.1. *On Popperian Parallel Learning*

PROPOSITION 3.1. $(2,3)$ PFIN $\not\subseteq (3,4)$ PFIN $\not\subseteq (4,5)$ PFIN.

Proof. Both noninclusions follow from winning strategies for Anke in the corresponding game $G(m, n; h, k)$.

The winning strategy of Anke for $G(2, 3; 3, 4)$ starts with creating three new leaves 11, 12, 13 below 1. Then Boris places his marker w.l.o.g. onto the leaf 11. Now Anke creates three new leaves below 2; Boris answers by moving to a node $2x$. The following diagram illustrates the situation, the first four rows show the positions of the four classes of Anke's markers $\{\mu_{D,j} : j \in D\}$ for $D=\{2, 3, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 4\}$, and $\{1, 2, 3\}$. The last row shows the positions of Boris' markers $v_1, ..., v_4$.

—	21	3	4
11	—	3	4
12	22	—	4
13	23	3	—
11	$2x$	3	4

If $2x=22$ then the game is in an A-configuration via $\{12, 23, 3, 4\}$. Otherwise the game is in an A-configuration via $\{13, 22, 3, 4\}$; thus Boris lost the game.

In this strategy Anke's second move depends on the first move of Boris. One can check that there is no winning strategy for Anke which is independent of Boris' moves.

In $G(3, 4; 4, 5)$ Anke's strategy is a little bit more complicated, but similar to the previous one. First Anke creates the leaves 11 and 12 and moves two of her markers to each of them; w.l.o.g. Boris moves from 1 to 11. Now Anke creates again two new leaves 111 and 112 and Boris follows w.l.o.g. from 11 to 111. Now Anke creates the new leaves 21 and 22; afterwards Boris follows from 2 to $2x$:

$$\begin{array}{ccccc}
- & 21 & 3 & 4 & 5 \\
111 & - & 3 & 4 & 5 \\
112 & 21 & - & 4 & 5 \\
12 & 22 & 3 & - & 5 \\
12 & 22 & 3 & 4 & - \\
\hline
111 & 2x & 3 & 4 & 5
\end{array}$$

This configuration is an A-configuration either via $\{112, 22, 3, 4, 5\}$ (if $2x = 21$) or via $\{12, 21, 3, 4, 5\}$ (if $2x = 22$). ▮

THEOREM 3.2. $(4, 5)$ PFIN $= (5, 6)$ PFIN.

Proof. It is sufficient to give Boris' winning strategy for the game $G(4, 5; 5, 6)$ since the other inclusion $(5, 6)$ PFIN $\subseteq$ $(4, 5)$ PFIN follows from Fact 1.2.

As long as Anke only splits nodes of the first subtree $j * \omega^*$, Boris always selects the path with the majority of the markers per leaf. As soon as Anke splits nodes of a second subtree, Boris moves according to one of the following five cases; w.l.o.g. we may assume the first subtree is $1 * \omega^*$ and the second subtree is $2 * \omega^*$. The diagrams below also indicate the strategy for all further moves.

Case (a). Assume that Boris' first marker shares its node only with one marker of Anke. Then there must have been three splittings generating four different nodes on the first subtree:

$$\begin{array}{cccccc}
- & * & * & * & * & * \\
a & - & w_1 & x_1 & y_1 & z_1 \\
b & u_1 & - & x_2 & y_2 & z_2 \\
c & u_2 & w_2 & - & y_3 & z_3 \\
d & u_3 & w_3 & x_3 & - & * \\
* & * & * & * & * & - \\
\hline
a & m_u & m_w & m_x & m_y & m_z
\end{array}$$

The letters in the matrix represent the current position of Anke's markers after her move. The letters in the last row show the countermoves of Boris in this situation. Here m_u denotes the majority of the nodes u_1, u_2, and u_3 if at least two of them are equal. The symbol $*$ means that the strategy does not depend on this entry in the diagram. For instance, marker v_1 simply follows marker $\mu_{\{1,2,3,4,5,6\},1}$, and marker v_2 follows the majority of the markers $\mu_{\{1,2,4,5,6\},2}$, $\mu_{\{1,2,3,5,6\},2}$, $\mu_{\{1,2,3,4,6\},2}$. In this way the diagram describes the complete strategy for Boris.

By way of contradiction assume that the game reaches an A-configuration via some set L after some move of Boris who played according to the above strategy. By the definition of A-configuration, at least four of the entries of each row of the matrix must belong to L. Thus the following must hold:

$$a \notin L \Rightarrow w_1, x_1, y_1, z_1 \in L;$$

$$b \notin L \Rightarrow u_1, x_2, y_2, z_2 \in L;$$

$$c \notin L \Rightarrow u_2, w_2, y_3, z_3 \in L;$$

$$d \notin L \Rightarrow u_3, w_3, x_3 \in L.$$

Since a, b, c, d are distinct nodes, at least three of them are not in L and, thus, two of the three entries u_1, u_2, u_3 are in L. There is only one leaf $\eta \geqslant 2$ in L and, thus, at least two of the three entries u_1, u_2, u_3 must be equal to this η, so $m_u = \eta \in L$. Similarly, m_w, m_x, m_y, $m_z \in L$. So five of Boris' markers are at positions in L; i.e., L is not an A-configuration. This contradiction proves that the above strategy wins for Boris.

Cases (b)–(e). The further case distinction depends on the form of the set $\mathcal{N}$ defined as $\mathcal{N} = \{(a, b), (a', b'), (a'', b''), (a''', b''')\}$, were $a, a', a'', a''' \geqslant 1$ and $b, b', b'', b''' \geqslant 2$ are the entries (marker positions) in the third, fourth, fifth, and sixth rows respectively:

$$\begin{array}{cccccc}
- & * & * & * & * & * \\
* & - & * & * & * & * \\
a & b & - & * & * & * \\
a' & b' & * & - & * & * \\
a'' & b'' & * & * & - & * \\
a''' & b''' & * & * & * & -
\end{array}$$

$\mathcal{N}$ contains up to four different pairs. Since Case (a) does not hold, Boris' first marker is placed on one of the nodes a, a', a'', a'''.

From now on, if two entries in a table are denoted by the same letter but have different indices, e.g., a_1 and a_2, then they are equal to a at the beginning, but may later become two different leaves.

Case (b). $\mathcal{N} = \{(a, b)\}$. Since the first and the second subtrees each have at least two leaves, the first two rows must have entries c and d with $c \neq a$ and $d \neq b$:

$$\begin{array}{cccccc}
- & d & w_1 & x_1 & y_1 & z_1 \\
c & - & w_2 & x_2 & y_2 & z_2 \\
a_1 & b_1 & - & x_3 & y_3 & z_3 \\
a_2 & b_2 & w_3 & - & * & * \\
* & * & * & * & - & * \\
* & * & * & * & * & - \\
\hline
a_1 & b_1 & m_w & m_x & m_y & m_z
\end{array}$$

Again m_w denotes the majority of the entries w_1, w_2, w_3, and so on. Consider a candidate L to witness an A-configuration. Similar to Case (a) the following implications hold:

$$\begin{aligned}
c \in L &\Rightarrow a_1, a_2 \notin L, b_1, b_2 \notin L, d \notin L \\
&\Rightarrow w_1, w_3, x_1, x_3, y_1, y_3, z_1, z_3 \in L \\
&\Rightarrow m_w, m_x, m_y, m_z \in L; \\
d \in L &\Rightarrow b_1, b_2 \notin L, a_1, a_2 \in L, c \notin L \\
&\Rightarrow w_2, w_3, x_2, x_3, y_2, y_3, z_2, z_3 \in L \\
&\Rightarrow m_w, m_x, m_y, m_z \in L; \\
c, d \notin L &\Rightarrow w_1, w_2, x_1, x_2, y_1, y_2, z_1, z_2 \in L \\
&\Rightarrow m_w, m_x, m_y, m_z \in L.
\end{aligned}$$

So in all three cases $m_w, m_x, m_y, m_z \in L$. Since, furthermore, either $a_1 \in L$ or $b_1 \in L$, it follows that five of Boris' markers are placed on nodes in L and thus Boris always moves into a B-configuration.

Case (c). All members of $\mathcal{N}$ have the same first component a, but the different second components b, c occur. Then the diagram w.l.o.g. looks like the one below, since there must be d different from a in the upper left corner:

$$\begin{array}{cccccc}
— & * & * & * & * & * \\
d & — & w & x & y & z \\
a_1 & b & — & * & * & * \\
a_2 & c & * & — & * & * \\
* & * & * & * & — & * \\
* & * & * & * & * & — \\
\hline
a_1 & b & w & x & y & z
\end{array}$$

Again consider a candidate L to witness an A-configuration. Since either $b \notin L$ or $c \notin L$, either $a_1 \in L$ or $a_2 \in L$. It follows that $d \notin L$ and $w, x, y, z \in L$. Furthermore, $a_1 \in L \vee b \in L$, so five of the entries a_1, b, w, x, y, z are in L and Boris always moves into a B-configuration.

Case (d). All members of $\mathcal{N}$ have in common the second component c. Boris applies the strategy obtained from the previous one by interchanging the role of the first and second columns:

$$\begin{array}{cccccc}
— & d & w & x & y & z \\
* & — & * & * & * & * \\
a & c_1 & — & * & * & * \\
b & c_2 & * & — & * & * \\
* & * & * & * & — & * \\
* & * & * & * & * & — \\
\hline
a & c_1 & w & x & y & z
\end{array}$$

For any candidate L, the properties $a \in L \vee c_1 \in L$ and $(a \notin L \vee b \notin L) \Rightarrow d \notin L \Rightarrow w, x, y, z \in L$ show, that Boris' strategy always gives a B-configuration.

Case (e). None of the cases above holds. There is $(a, b) \in \mathcal{N}$ such that Boris' first marker is placed on a. Either there is now a pair $(c, d) \in \mathcal{N}$ which differs from (a, b) in both components or there are $(a, d), (c, b) \in \mathcal{N}$ which differ from the pair (a, b) in one component. The second subcase reduces to the first by considering (a, d) instead of (a, b), since (c, b) differs from (a, d) in both components. For the first subcase, Boris' winning strategy is given by the following diagram:

$$\begin{array}{cccccc}
— & * & * & * & * & * \\
* & — & * & * & * & * \\
a & b & — & x & y & z \\
c & d & w & — & * & * \\
* & * & * & * & — & * \\
* & * & * & * & * & — \\
\hline
a & b & w & x & y & z
\end{array}$$

For any candidate L, one of the entries a, b and one of the entries c, d must be in L. Now $a \notin L \vee b \notin L$ and $c \notin L \vee d \notin L$ follow from $a \notin L \vee c \notin L$ and $b \notin L \vee d \notin L$. So $w \in L$ by $c \notin L \vee d \notin L$ and $x, y, z \in L$ by $a \notin L \vee b \notin L$. Again five of the entries a, b, w, x, y, z are always in L. So Boris wins the game in all cases. ∎

In the game $G(n, n+1; n+1, n+2)$ for $n < 4$, the proof does not work since then Case (a) breaks down (as witnessed by Proposition 3.1). But for $n > 4$ the above proof can be generalized; in fact we get $(\forall n > 4)\ [(4, 5)\ \text{PFIN} = (n, n+1)\ \text{PFIN}]$.

In the game $G''(n, n+1; n+1, n+2)$, Boris can keep all markers at level 1 until Anke has split at least two of the starting nodes $1, ..., k$. Then Boris moves his first marker so that he avoids Case (a) and employs the winning strategy given by the other cases, which also goes through for $n = 2, 3$. Thus $(2, 3)\ \text{PFIN} \subseteq (3, 4)\ \text{FIN}$ and $(3, 4)\ \text{PFIN} \subseteq (4, 5)\ \text{FIN}$.

Open Problem. Find an explicit characterization of the equality problem for PFIN, i.e., of the set $\{(m, n; h, k): (m, n)\ \text{PFIN} = (h, k)\ \text{PFIN}\}$.

3.2. *On Parallel Learning*

Since the condition of Theorem 2.5 is not a characterization as in Theorem 2.3, the following Proposition 3.3 must be proved in a direct way.

PROPOSITION 3.3. *If* $(m, n)\ \text{FIN} \not\subseteq (h, k)\ \text{FIN}$ *then* $(m, n+1)\ \text{FIN} \not\subseteq (h, k+1)\ \text{FIN}$.

Proof. Let $S' \in (m, n)\ \text{FIN} - (h, k)\ \text{FIN}$, w.l.o.g. $f(0) = 0$ for all $f \in S$. A set $S = S' \cup \{g_i : i \in \omega\} \in (m, n+1)\ \text{FIN} - (h, k+1)\ \text{FIN}$ is constructed via a sequence of functions g_i such that g_i are of the form $e_i 0^{a_i} b_i 0^{\omega}$, where $e_0 = 1$, $e_{i+1} = e_i + a_i + 2$, and $a_i, b_i \in \omega$.

These conditions already guarantee that $S \in (m, n+1)$ FIN: Given $n+1$ functions ordered by the first value ($f_1(0) \leqslant f_2(0) \leqslant \cdots \leqslant f_{n+1}(0)$) there is a u such that $0 = f_u(0) < f_{u+1}(0)$, w.l.o.g. $u \leqslant n$. The indices of $f_{u+1}, ..., f_n$ can be calculated from the initial segments of length $f_{n+1}(0)$. So one obtains $n-u$ correct indices. Since $f_1, ..., f_u \in S'$, and, by Fact 1.3, $S' \in (m-(n-u), n-(n-u))$ FIN, u indices can be calculated such that $m-(n-u)$ of them are correct. In total there are n indices for $f_1, ..., f_n$ of which m are correct. Thus, $S' \in (m, n+1)$ FIN.

It is possible to diagonalize against the ith $(h, k+1)$ PFIN-machine M_i while defining g_i. Since $S' \notin (h, k)$ FIN, there are $f_1, ..., f_k \in S'$. such that

- Either, M_i does not converge on input $f_1, ..., f_k, e_i 0^\omega$. Then let $g_i = e_i 0^\omega$, $a_i = 0$, $b_i = 0$, and $e_{i+1} = e_i + 2$.
- Or, M_i converges after reading a_i arguments to $k+1$ indices such that $k+1-h$ of the indices for $f_1, ..., f_k$ are incorrect. Then select b_i such that the index for $g_i = e_i 0^{a_i} b_i 0^\omega$ is also incorrect and let $e_{i+1} = e_i + a_i + 2$.

In both cases g_i is selected as a witness against M_i (together with $f_1, ..., f_k$). So $S \notin (h, k+1)$ FIN. ∎

In the following we show explicit noninclusions by providing winning strategies for Anke in $G'(m, n; h, k)$.

PROPOSITION 3.4. (2, 3) FIN $\not\subseteq$ (3, 4) FIN.

Proof. The first move of Anke's winning strategy for the game $G'(2, 3; 3, 4)$ creates an A-configuration via the set $\{11, 21, 3, 4\}$ in order to force Boris to move at least one marker, w.l.o.g. Boris moves his first marker:

—	2	3	4
1	—	3	4
11	2	—	4
1	21	3	—
11	2	3	4

Now Anke moves the markers, which remained on 1, to the node 12 and those, which remained on 2, to the node 21. Now the set $\{12, 21, 4\}$ witnesses an A-configuration, since it contains two markers of each row, but it does not contain three of Boris' markers.

—	21	3	4
12	—	3	4
11	21	—	4
12	21	3	—
11	21	3	4

Boris can move his second marker to 21, but he cannot move his marker from 11 to 12. So $\{12, 21, 4\}$ contains at most two of Boris' markers and thus he has lost the game. ∎

PROPOSITION 3.5. $(m, n+1)$ FIN $\not\subseteq$ $(n+1, n+2)$ FIN.

Proof. Anke's winning strategy for this game goes through a loop up to n times.

The loop invariant before the jth iteration ($j = 1, ..., n$):

- All markers in the ith column are on node $i * 1$ for $i < j$;
- All markers in the ith column are on node i for $i = j, ..., n+1$;
- $j-1$ of Anke's markers are on node $(n+2) * 1^{j-2}$;
- $n+2-j$ of Anke's markers and Boris' last marker are on $(n+2) * 1^{j-1}$.

So the main idea of this loop is to isolate more and more Boris' last marker; either Boris will once try to escape from this loop and lose within one move or Boris will lose after all n iterations of the loop.

The diagram shows the first, $(j-1)$th, jth, $(j+1)$th, $(n+1)$th, and $(n+2)$th rows; note that the rows and columns indexed by $1, ..., j-1$ only exist for $j > 1$.

—	$\cdots$	$(j-1)*1$	j	$j+1$	$\cdots$	$n+1$	$(n+2)*1^{j-2}$
$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$1*1$	$\cdots$	—	j	$j+1$	$\cdots$	$n+1$	$(n+2)*1^{j-2}$
$1*1$	$\cdots$	$(j-1)*1$	—	$j+1$	$\cdots$	$n+1$	$(n+2)*1^{j-1}$
$1*1$	$\cdots$	$(j-1)*1$	j	—	$\cdots$	$n+1$	$(n+2)*1^{j-1}$
$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$1*1$	$\cdots$	$(j-1)*1$	j	$j+1$	$\cdots$	—	$(n+2)*1^{j-1}$
$1*1$	$\cdots$	$(j-1)*1$	j	$j+1$	$\cdots$	$n+1$	—
$1*1$	$\cdots$	$(j-1)*1$	j	$j+1$	$\cdots$	$n+1$	$(n+2)*1^{j-1}$

Anke's move in the jth iteration. Anke moves her markers as follows:

- In column j and rows $1, ..., j-1$ from j to $j * 1$;
- In column $n+2$ and rows $j+1, ..., n+1$ from $(n+2) * 1^{j-1}$ to $(n+2) * 1^j$.

Now $\{1*1, ..., (j-1)*1, j*1, j+1, ..., n+1, (n+2)*1^j\}$ is a witness for an A-configuration. So Boris has to react: Either he moves his jth marker from j to $j*1$ or his last marker from $(n+2) * 1^{j-1}$ to $(n+2) * 1^j$; in the first iteration Boris takes the second possibility since $\{2, 3, ..., n+1, (n+2) * 1\}$ is also a witness for the A-configuration.

If Boris moves from j to $j * 1$. Anke wins the game by moving her markers as follows:

- In column j and rows $j+1, ..., n+2$ from j to $j * 2$;
- In column $n+2$ and rows $1, ..., j-1$ from $(n+2) * 1^{j-2}$ to $(n+2) * 1^{j-2} * 2$.

Now $L = \{1*1, ..., (j-1)*1, j*2, j+1, ..., n+1, (n+2) * 1^{j-2} * 2\}$ witnesses an A-configuration. But since Boris' jth marker is on $j * 1$ and his last marker is on $(n+2) * 1^{j-1}$, he cannot move any of these markers to a node in L, so he cannot reach a B-configuration:

$$\begin{array}{cccccccc}
\text{—} & \cdots & (j-1)*1 & j*1 & j+1 & \cdots & n+1 & (n+2)*1^{j-2}*2 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1*1 & \cdots & \text{—} & j*1 & j+1 & \cdots & n+1 & (n+2)*1^{j-2}*2 \\
1*1 & \cdots & (j-1)*1 & \text{—} & j+1 & \cdots & n+1 & (n+2)*1^{j-1} \\
1*1 & \cdots & (j-1)*1 & j*2 & \text{—} & \cdots & n+1 & (n+2)*1^{j} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1*1 & \cdots & (j-1)*1 & j*2 & j+1 & \cdots & \text{—} & (n+2)*1^{j} \\
1*1 & \cdots & (j-1)*1 & j*2 & j+1 & \cdots & n+1 & \text{—} \\
\hline
1*1 & \cdots & (j-1)*1 & j*1 & j+1 & \cdots & n+1 & (n+2)*1^{j-1}
\end{array}$$

Otherwise; *Boris moves from* $(n+2)*1^{j-1}$ *to* $(n+2)*1^{j}$. Now Anke and Boris adjust their markers before the start of the next iteration:

- Anke moves in column j and rows $j+1, ..., n+2$ from j to $j*1$ and in column $n+2$ and rows $1, ..., j-1$ from $(n+2)*1^{j-2}$ to $(n+2)*1^{j-1}$;
- Boris moves in column j from j to $j*1$ (this move is safe since all markers of Anke have moved from j to $j*1$):

$$\begin{array}{cccccccc}
\text{—} & \cdots & (j-1)*1 & j*1 & j+1 & \cdots & n+1 & (n+2)*1^{j-1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1*1 & \cdots & \text{—} & j*1 & j+1 & \cdots & n+1 & (n+2)*1^{j-1} \\
1*1 & \cdots & (j-1)*1 & \text{—} & j+1 & \cdots & n+1 & (n+2)*1^{j-1} \\
1*1 & \cdots & (j-1)*1 & j*1 & \text{—} & \cdots & n+1 & (n+2)*1^{j} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1*1 & \cdots & (j-1)*1 & j*1 & j+1 & \cdots & \text{—} & (n+2)*1^{j} \\
1*1 & \cdots & (j-1)*1 & j*1 & j+1 & \cdots & n+1 & \text{—} \\
\hline
1*1 & \cdots & (j-1)*1 & j*1 & j+1 & \cdots & n+1 & (n+2)*1^{j}
\end{array}$$

This is just the situation at the beginning of the next iteration of the loop. Thus the algorithm continues there.

After all n iterations of the loop. The algorithm reaches this step only if Boris during all iterations selected the "otherwise" branch. Now Boris' marker shares his position with only one of Anke's markers. Then Anke moves all markers from $(n+2)*1^{n-1}$ to $(n+2)*1^{n-1}*2$ and the set $L=\{1*1, 2*1, ..., (n-1)*1, n*1, (n+2)*1^{n-1}*2\}$ witnesses that the game is in an A-configuration, the set contains only Boris' first n markers. Of course, Boris' cannot move his $(n+1)$-st marker to a node in L. Boris' $(n+2)$-nd marker is on node $(n+2)*1^{n}$ which is incomparable to $(n+2)*1^{n-1}*2$. Thus Boris cannot move into a B-configuration:

$$\begin{array}{ccccccc}
\text{—} & 2*1 & \cdots & (n-1)*1 & n*1 & n+1 & (n+2)*1^{n-1}*2 \\
1*1 & \text{—} & \cdots & (n-1)*1 & n*1 & n+1 & (n+2)*1^{n-1}*2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
1*1 & 2*1 & \cdots & \text{—} & n*1 & n+1 & (n+2)*1^{n-1}*2 \\
1*1 & 2*1 & \cdots & (n-1)*1 & \text{—} & n+1 & (n+2)*1^{n-1}*2 \\
1*1 & 2*1 & \cdots & (n-1)*1 & n*1 & \text{—} & (n+2)*1^{n} \\
1*1 & 2*1 & \cdots & (n-1)*1 & n*1 & n+1 & \text{—} \\
\hline
1*1 & 2*1 & \cdots & (n-1)*1 & n*1 & n+1 & (n+2)*1^{n}
\end{array}$$

So Anke wins the game. ∎

Propositions 3.3 and 3.5 imply that (m, n) FIN $\nsubseteq (m+1, n+1)$ FIN for all m, n with $1 \leqslant m < n$. This confirms a conjecture of Kinber and Wiehagen [12]. They already indicated in [12, p. 15] that their conjecture implies that there are no nontrivial equalities between the FIN-classes.

THEOREM 3.6. (m, n) FIN $= (h, k)$ FIN $\Leftrightarrow (m=h \wedge n=k) \vee (m=n \wedge h=k)$.

Proof. The if direction is trivial. For the converse assume that (m, n) FIN $= (h, k)$ FIN and, say, $n \leqslant k$. By Fact 1.3 it follows that $n-m=k-h$. Thus, if $n=k$ then $m=h$.

Now assume that $n<k$. We show that $m=n$ (and, therefore, $h=k$). By Fact 1.2 we get (h, k) FIN $\subseteq (h-1, k-1)$ FIN $\subseteq \cdots \subseteq (h-b, k-b)$ FIN, for every $b<h$. Thus, for $b=k-n-1$, (h, k) FIN $\subseteq (m+1, n+1)$ FIN; i.e., (m, n) FIN $\subseteq (m+1, n+1)$ FIN. As we noted above, this implies $m=n$. ∎

Together with the facts $(3, 4)$ PFIN $\nsubseteq (4, 5)$ PFIN, $(4, 5)$ PFIN $= (5, 6)$ PFIN, and $(3, 4)$ PFIN $\subseteq (4, 5)$ FIN, Proposition 3.5 shows that all three inclusion problems are different. (The fourth type of inclusion (FIN versus PFIN) is not interesting since it never holds: $(\forall n)$ [FIN $\nsubseteq (1, n)$ PFIN]. This is witnessed by the family $S=\{f \in \mathrm{REC}: \varphi_{f(0)}=f\}$.)

COROLLARY 3.7. *The following three inclusion relations are pairwise different*:

- $\{(m, n, h, k): (m, n)$ PFIN $\subseteq (h, k)$ PFIN$\}$;
- $\{(m, n, h, k): (m, n)$ PFIN $\subseteq (h, k)$ FIN$\}$;
- $\{(m, n, h, k): (m, n)$ FIN $\subseteq (h, k)$ FIN$\}$.

3.3. *Admissible Sets*

Frequency computation was first studied by Rose [23] and Trakhtenbrot [25]; see [8] for a recent survey. A function $f: \omega \rightarrow \omega$ is called (m, n)-*recursive* iff there is a recursive function $G: \omega^{n} \rightarrow \omega^{n}$ such that for all $x_1 < \cdots < x_n$, $G(x_1, ..., x_n)$ and $(f(x_1), ..., f(x_n))$ agree in at least m components. The inclusion problem for frequency computation is the question for which m, n, h, k, every (m, n)-recursive function is (h, k)-recursive. A combinatorial characterization which implies that the inclusion problem is decidable was recently obtained in [16, 19].

To study the inclusion problem, Degtev [5] introduced the notion of "(m, n)-admissible sets" which formalizes a finite combinatorial version of (m, n)-computation. (It also appeared implicitly in Kinber's thesis [10].) We show that this notion is also useful for the study of parallel learning, since it leads to further explicit noninclusions. This is not surprising, because the notion of parallel learning is a learning theoretic counterpart of frequency computation.

DEFINITION 3.8. Let $s \geqslant n \geqslant m \geqslant 1$. A finite set $V \subseteq \omega^s$ is called (m, n)-*admissible* iff for every n numbers x_i $(1 \leqslant x_1 < \cdots < x_n \leqslant s)$ there exists a vector $(b_1, ..., b_n) \in \omega^n$ such that for every $v \in V$

$$|\{i: v[x_i] = b_i\}| \geqslant m.$$

In other words, there exists a function $G: \{1, ..., s\}^n \to \omega^n$ such that for all pairwise distinct $x_1, ..., x_n \in \{1, ..., s\}$, $|\{i: v[x_i] = (G(x_1, ..., x_n))_i\}| \geqslant m$.

It is easy to see that the corresponding inclusions and noninclusions of Facts 1.2, 1.3 also hold for admissible sets. For instance, every $(m+1, n+1)$-admissible set is (m, n)-admissible, and, for $n - m > k - h$, the set $\{0, 1\}^{n-m} \times \{0\}^{\max(n, k)}$ is (m, n)-admissible but not (h, k)-admissible.

THEOREM 3.9. *If V is (m, n)-admissible, but not (h, k)-admissible, then* (m, n) PFIN $\not\subseteq (h, k)$ FIN; *in particular*, (m, n) FIN $\not\subseteq (h, k)$ FIN *and* (m, n) PFIN $\not\subseteq (h, k)$ PFIN.

Proof. If $k < n$, then an (m, n)-admissible set V which is not (h, k)-admissible exists only for $n - m > k - h$ and so Theorem 3.9 reduces to Fact 1.3.

Let $n \leqslant k$ and let $V \subseteq \{1, ..., q\}^k$ be (m, n)-admissible but not (h, k)-admissible. By the remark following Theorem 2.5 it suffices to show that Anke has a winning strategy in the game $G''(m, n; h, k)$.

In the first move, Anke places her markers on the leaves according to an (m, n)-operator for V; i.e., if the (m, n)-operator for $D = \{i_1, ..., i_n\}$ gives $(b_1, ..., b_n)$ then each marker μ_{D, i_j} is placed on the leaf $i_j * b_j$. Thus for every $v \in V$ the associated set $L_v = \{i * v[i]: 1 \leqslant i \leqslant k\}$ witnesses an A-configuration.

Assume that Boris could move into a B-configuration by placing his markers on nodes $1 * c_1, 2 * c_2, ..., k * c_k$. Then for each v, h markers are in the set L_v and h components of $(c_1, c_2, ..., c_k)$ agree with the corresponding components of v. Thus V would be (h, k)-admissible via $(c_1, c_2, ..., c_k)$ a contradiction. Thus whatever Boris does, the game remains in an A-configuration and Anke wins the game. ∎

For example, the set $\{1^k, 2^k, ..., n^k\}$ is $(1, n)$-admissible but not (h, k)-admissible for any h, k with $h/k > 1/n$. Thus, we get the following noninclusion.

COROLLARY 3.10. *If* $1/n < h/k$ *then* $[(1, n)$ FIN $\not\subseteq (h, k)$ FIN $\wedge$ $(1, n)$ PFIN $\not\subseteq (h, k)$ PFIN$]$.

Further noninclusions can be derived from the following fact.

FACT 3.11 [14, Lemma 9.5]. *If one of the following conditions holds then there is an (m, n)-admissible set V which is not (h, k)-admissible*:

(a) $n - 2m > k - 2h \geqslant 0$;

(b) $n = 2m + 1$, $k = 2h + 1$, *and* $k > n$;

(c) *There is $(m, n-1)$-admissible set W which is not $(h, k-1)$-admissible.*

Proof. (a) Let $V = \{0, 1\}^{n-2m} \times \{0^{k+n}, 1^{k+n}\}$.

(b) Let V contain 0^k, 1^k, all vectors $0^i 1 0^{k-i-1}$ for $i = 0, ..., k-1$, $10^{k-2}1$, and $0^i 110^{k-i-2}$ for $i = 0, ..., k-2$. Note that V is the closure of $\{0^k, 10^{k-1}, 110^{k-2}, 1^k\}$ under "rotational shifts."

(c) Let $V = W \times \{0, 1\}$.

See [14] for the verification that the sets V have the required properties. ∎

FACT 3.12 [10, Theorem 1.6]. *Every $(n, n+1)$-admissible set is $(n+1, n+2)$-admissible for $n \geqslant 2$.*

Proof. It is sufficient to show this for subsets $V \subseteq \omega^{n+2}$. Let V be $(n, n+1)$-admissible $(n \geqslant 2)$ and let $0^{n+2} \in V$. If V is not $(n+1, n+2)$-admissible via 0^{n+2} then some vector has two nonzero components, say, $1^2 0^n \in V$. Since V is $(1, 2)$-admissible, there are a and b such that $v[0] = a$ or $v[1] = b$ for every $v \in V$, say, $a = 1$ and $b = 0$. Then V is $(n+1, n+2)$-admissible via 10^{n+1}. Otherwise, there would exist some $v \in V$ differing in two components from 10^{n+1}. Since $v[0] = 1$ or $v[1] = 0$, it follows that v differs either from 0^{n+2} or from $1^2 0^n$ in three components. Thus V is not $(n, n+1)$-admissible, a contradiction. ∎

So on one hand, $(2, 3)$ FIN $\not\subseteq (3, 4)$ FIN and $(2, 3)$ PFIN $\not\subseteq (3, 4)$ PFIN and, on the other hand, every $(2, 3)$-admissible set is $(3, 4)$-admissible. Thus the admissibility criterion does not characterize these inclusion relations.

COROLLARY 3.13. *The inclusion relations of* FIN *and* PFIN *both differ from the admissibility criterion.*

Nevertheless results on admissible sets allow a further partial result for the equality problem of PFIN.

PROPOSITION 3.14. *If $m/n < 2/3$ and $(h, k) \neq (m, n)$ then there exists either an (m, n)-admissible set which is not (h, k)-admissible or an (h, k)-admissible set which is not (m, n)-admissible.*

Proof. If $n-m>k-h$ then the set of all binary vectors with up to $n-m$ ones is (m, n)-admissible but not (h, k)-admissible; thus the case $n-m=k-h$ remains, w.l.o.g. $n<k$. If an (m, n)-admissible set is (h, k)-admissible it is also $(m+1, n+1)$-admissible thus it is sufficient to give (m, n)-admissible sets which are not $(m+1, n+1)$-admissible. We distinguish two cases:

- m is odd; i.e., $n=s+3r-1$, $m=2r-1$. Then

$$V=\{1^{2r-1}0^{r+1}, 0^{3r}, 0^{3r-1}1, 0^i10^{2r-2}10^{r-i}: i=0, 1, ..., r-1\} \times \{0, 1\}^s.$$

- m is even, i.e., $n=s+3r-2$, $n=2r-2$. Then

$$V=\{1^{2r-1}0^r, 0^{3r-1}, 0^{3r-2}1, 0^r10^{2r-2}, 0^i10^{2r-2}10^{r-i-1}: i=0, 1, ..., r-1\} \times \{0, 1\}^s.$$

A first example for a $(3, 5)$-admissible set which is not $(4, 6)$-admissible is due to Kinber [10]. ▮

COROLLARY 3.15. *If* $m/n<2/3$ *and* $(h, k) \neq (m, n)$ *then* (m, n) PFIN $\neq (h, k)$ PFIN.

4. ORACLES FOR FINITARY GAMES

In Definition 2.1 we have introduced the notion of a finite game $\mathscr{G}=(G_1, G_2, W, s_0, t_0)$ in order to characterize the inclusion problem for PFIN. Our next goal is to determine when (m, n) PFIN $\subseteq (h, k)$ PFIN$[A]$. Here (h, k) PFIN$[A]$ is the class of all $S \in$ REC which are (h, k) PFIN-inferable by an algorithm which has access to oracle $A \subseteq \omega$. To this end we have to investigate the "off-line" version of $G(m, n; h, k)$. But this is only a special case of a more general approach which works for arbitrary finite games $\mathscr{G}$ and which may be of use in similar situations and for other inference criteria. In this section we study the general approach and in the next section we discuss the application to PFIN. Some of the ideas in this section have previously been used in [13].

DEFINITION 4.1. In the *off-line* version of $\mathscr{G}$, Anke announces at the beginning the list of her moves $(v_1, ..., v_k)$ in rounds $1, ..., k$. Here, v_{i+1} must be properly adjacent to v_i for $i=0, ..., k-1$ (where $v_0=s_0$) and v_k must not have outcoming edges. Boris wins iff there is a list of counter moves $(w_0, ..., w_k)$ such that Boris wins the original game if both players play according to their move list; i.e., Boris moves from t_0 to w_0, Anke from s_0 to v_1, Boris from w_0 to w_1, ... until Anke moves from v_{k-1} to v_k and Boris wins the game by his last move from w_{k-1} to w_k. Formally, w_0 is adjacent to t_0, w_{i+1} is adjacent to w_i for $i=0, ..., k-1$ and $(v_i, w_i) \in W$ for $i=0, ..., k$.

In the *infinite* version of $\mathscr{G}$ both players are allowed to perform empty moves and we drop the condition that the position after each move of Boris belongs to W. There are ω many rounds. Since G_1, G_2 are finite and acyclic it follows that at almost all rounds the marker of Anke [Boris] is at some fixed node s_1 [t_1]. Boris wins the game iff $(s_1, t_1) \in W$. It is easy to see that any winning strategy for the finite version can be translated into a winning strategy for the infinite version.

We are interested in questions of computability for the *off-line version of the infinite game*. Suppose we are given an index i for the list of moves of Anke in the infinite game, i.e., φ_i is total and $\varphi_i(0)=s_0$ and $\varphi_i(n)$ is the position of Anke's marker at the end of round n. Can we compute uniformly in i a list of counter moves for Boris such that he wins the resulting infinite game? We want to characterize the oracles A such that this can be done recursive in A. Let comp$(\mathscr{G})$ denote the class of all such oracles A.

Let PA denote the class of all degrees containing a complete and consistent extension of *Peano arithmetic*. See [20, pp. 510–515] for background information. It is known that PA contains low degrees, in particular, degrees which are strictly below the degree of the halting problem K. Let $DNR_k=\{d: \omega \to \{0, ..., k-1\} \mid (\forall i) \quad [d(i) \neq \varphi_i(i)]\}$. Jockusch [9, Proposition 2] showed that PA coincides with the degrees of functions in DNR_k for all $k \geqslant 2$.

THEOREM 4.2. *There are exactly four possible cases*:

(1) *If Boris has a winning strategy for* $\mathscr{G}$ *then* comp$(\mathscr{G})=\{A: A \subseteq \omega\}$.

(2) *If Boris has a winning strategy for the off-line version of* $\mathscr{G}$ *but not for* $\mathscr{G}$, *then* comp$(\mathscr{G})=\{A: dg_T(A) \in PA\}$.

(3) *If Anke has a winning strategy for the off-line version and for every s adjacent to* s_0 *there is t adjacent to* t_0 *with* $(s, t) \in W$, *then* comp$(\mathscr{G})=\{A: A \geqslant_T K\}$.

(4) *If there is s adjacent to* s_0 *such that* $(s, t) \notin W$ *for all t adjacent to* t_0, *then* comp$(\mathscr{G})=\varnothing$.

Proof. (1) and (4) are obvious.

(2) The proof has two parts. In the first part we show that if Boris has winning strategy for the off-line version of $\mathscr{G}$, then comp$(\mathscr{G}) \supseteq \{A: dg_T(A) \in PA\}$. In the second part we show that if Anke has a winning strategy for $\mathscr{G}$, then comp$(\mathscr{G}) \subseteq \{A: dg_T(A) \in PA\}$.

First part. Assume that Boris has a winning strategy for the (finite) off-line version of $\mathscr{G}$. Every list of counter moves $(w_0, ..., w_k)$ induces in a uniform way a list of counter moves for the infinite off-line version as we now explain.

Suppose we are given an index i for the list of moves for Anke. We define h_i, the induced list of counter moves, as follows:

Let $h_i(0)=w_0$. If $h_i(n)=w_m$, $(\varphi_i(n+1), h_i(n)) \notin W$, and $m<k$, then let $h_i(n+1)=w_{m+1}$; else let $h_i(n+1)=h_i(n)$.

We say that $w = (w_0, ..., w_k)$ loses against i in step n if n is minimal such that $(\varphi_i(n), h_i(n)) \notin W$. In that case we write $l(w, i) = n$. If n does not exist then $l(w, i) = \infty$. Note that the graph of $l(-, -)$ is uniformly recursive (assuming that the second component is an index of a total recursive function).

If $l(w, i) = \infty$ then, in particular, the induced h_i wins against φ_i in the infinite version of the game. It easily follows from the hypothesis that for every infinite list of moves φ_i of Anke, there exists w with $l(w, i) = \infty$.

Since the off-line version of the finite game has at most $k = |V_1|$ rounds, we may assume that all lists of counter moves have length k. Let L be the finite set of all these lists.

Now suppose that we are given an index i of the list of moves of Anke in the infinite game. We show that if $dg_T(A) \in \mathrm{PA}$ then we can A-recursively compute a finite list w which does not lose against i in any step. By the remarks above, this completes the proof of the first part.

By the hypothesis we know that a suitable w is contained in L. So it suffices to provide an A-recursive reduction procedure which reduces L to a one-element set that still contains a suitable w.

Construction. As long as $|L| > 1$ choose different lists $u, w \in L$ and compute an index e of the following constant function f:

$$f(x) = \begin{cases} 0, & \text{if } l(u, i) < \infty \wedge l(u, i) \leqslant l(w, i); \\ 1, & \text{if } l(w, i) < l(u, i); \\ \uparrow, & \text{otherwise.} \end{cases}$$

Since $dg_T(A) \in \mathrm{PA}$ there is $d \leqslant_T A$ such that $d \in DNR_2$. Thus, we can A-recursively *exclude* either $\varphi_e(e) = 0$ (if $d(e) = 0$) or $\varphi_e(e) = 1$ (if $d(e) = 1$). In the first case we let $L = L - \{w\}$, else we let $L = L - \{u\}$. Then we repeat the procedure.

End of construction.

Note that if the list which we remove does not lose against i at any step, then the list that we keep in L has the same property. Thus at each step L contains a suitable list; i.e., the reduction procedure is correct. This completes the proof of the first part.

Second part. Assume that Anke has a winning strategy for $\mathcal{G}$. In our case this is just a function $p: V_1 \times V_2 \to V_1$ such that Anke wins if she plays $p(v_1, v_2)$ in every position (v_1, v_2) where it is her turn to move.

> We may assume w.l.o.g. that in every position $(v_1, v_2) \in W$ which is reachable when Anke plays according to her winning strategy, we have $(p(v_1, v_2), v_2) \notin W$. $(*)$

Let $a = |V_1|$, $V_2 = \{w_0, ..., w_{k-1}\}$. Suppose that $A \in \mathrm{comp}(\mathcal{G})$. We shall show that there is an A-recursive function in DNR_k. As was mentioned above this implies $dg_T(A) \in \mathrm{PA}$.

To this end we define inductively for every sequence of a numbers $\sigma = (z_1, ..., z_a)$ a move list $g = g_\sigma$ for Anke in the infinite off-line version of $\mathcal{G}$ as follows.

Construction.
Initialization. Let $n = 0$; $g(0) = s_0$; $v = s_0$; $w = t_0$. Go to step 1.
Step j. Let $C_j = \{i: w_i \text{ adjacent to } w \text{ and } (v, w_i) \in W\}$.
While $\varphi_{z_j, n}(z_j) \notin C_j$ let $g(n+1) = g(n)$ and let $n = n+1$.
If $\varphi_{z_j, n}(z_j) \in C_j$ then let $w = w_i$ for $i = \varphi_{z_j, n}(z_j)$, let $g(n+1) = p(v, w)$, $v = p(v, w)$, $n = n+1$, and go to step $j+1$.
End of construction.

Note that g is the sequence of moves according to the winning strategy of Anke against a potential Boris who chooses his moves in round j as follows: He waits until $\varphi_{z_j}(z_j)$ is defined, say equal to i. Then he moves to w_i (if this is correct and produces a position in W).

Hence, any A-recursive counterstrategy that wins against g must be different from this potential strategy. We complete the proof by showing that if one can A-recursively compute different counter strategies for all such g, then $dg_T(A) \in \mathrm{PA}$.

By the hypothesis, there exists in a uniform way an A-recursive infinite list f_σ of countermoves for Boris with $(s_1, t_1) \in W$ for $s_1 = \lim_n g_\sigma(n)$ and $t_1 = \lim_n f_\sigma(n)$. We may assume w.l.o.g.:

$$[g_\sigma(n+1) = g_\sigma(n) \wedge (g_\sigma(n), f_\sigma(n)) \in W] \Rightarrow f_\sigma(n+1) = f_\sigma(n). \qquad (**)$$

Let $n_j(\sigma)$ denote the jth number n (in increasing order) such that $g_\sigma(n+1) \neq g_\sigma(n)$, if it exists. For every $\sigma = (z_1, ..., z_a)$ and every i $(1 \leqslant i \leqslant a)$ we define a predicate $P(i, \sigma)$ as

$$P(i, \sigma) \Leftrightarrow (\forall j, 1 \leqslant j < i) [n_j(\sigma)\downarrow \wedge f_\sigma(n_j(\sigma)) = w_m \text{ for } m = \varphi_{z_j(z_j)}].$$

Note that trivially $P(1, \sigma) \equiv$ true. Also note that $P(i, \sigma)$ is r.e. in A. Intuitively, if $P(i, \sigma)$ holds then g_σ has correctly predicted the behavior of f_σ up to round i.

If g_σ would correctly predict up to round a then $(g(n_{a-1}+1), f(n_{a-1}))$ would be a final position in $\mathcal{G}$ such that $(g(n_{a-1}+1), w) \notin W$ for any node w adjacent to $f(n_{a-1})$. Furthermore, $g(n) = g(n_{a-1}+1)$ for all $n > n_{a-1}$. Since $\lim_n f(n)$ is adjacent to $f(n_{a-1})$ we would have $(\lim_n g(n), \lim_n f(n)) \notin W$, contradicting the property of f. Therefore, $P(a, \sigma) \equiv$ false. Consider the least i with $1 \leqslant i < a$ such that

$$(\exists z_{i+1}, ..., z_a)(\forall z_1, ..., z_i)[\neg P(i+1, \sigma) \text{ for } \sigma = (z_1, ..., z_a)].$$

Note that i exists because $\neg P(a, \sigma) \equiv$ true. For the following we fix witnesses $z_{i+1}, ..., z_a$. If $i > 1$ then using the minimality of i, we get

$$\neg(\exists z_i) \quad (\forall z_i, ..., z_{i-1}) \quad [\neg P(i, \sigma)].$$

Or, equivalently,

$$(\forall z_i) \quad (\exists z_1, ..., z_{i-1}) \quad [P(i, \sigma)]. \qquad (+)$$

For $i = 1$ this holds trivially since $P(1, \sigma) \equiv$ true. Now we can A-recursively compute a function $d \in DNR_k$ as follows.

Construction. On input z_i we search for $z_1, ..., z_{i-1}$ such that $P(i, \sigma)$ holds. The search is effective since $P(-, -)$ is r.e. in A. By $(+)$, the search terminates.

Let $n_{i-1} = n_{i-1}(\sigma)$, $f = f_\sigma$, $g = g_\sigma$. By the choice of $z_{i+1}, ..., z_a$ we know that $P(i+1, \sigma)$ does not hold. This means:

$$\text{If } n_i \downarrow \wedge (g(n_i), f(n_i)) \in W, \text{ then there is } m \text{ with } f(n_i) = w_m \wedge m \neq \varphi_{z_i}(z_i). \qquad (++)$$

Therefore we search for the least $n' > n_{i-1}$ such that

(a) $n' = n_i$, or

(b) $(g(n'), f(n')) \in W$.

If the search terminates by (a) then we know $\varphi_{z_i}(z_i)$ and define $d(z_i) = \min\{x : x \neq \varphi_{z_i}(z_i)\}$. If the search terminates by (b) then we let $d(z_i) = m$ with $w_m = f(n')$.

End of construction.

Clearly $d(z_i) < k$. By the property of f we have $(g(n), f(n)) \in W$ for all sufficiently large n. Thus the search terminates and d is total.

If n_i is undefined and $\varphi_{z_i}(z_i) = m'$ then $(g(n'), w_{m'}) \notin W$ or $w_{m'}$ is not adjacent to $f(n_{i-1})$. Hence, in this case $m \neq m'$.

Now suppose that the search terminates by (b) and n_i is defined. Then $n_i > n'$. Since $g(n) = g(n')$ for $n' \leqslant n \leqslant n_i$ we get by assumption $(**)$ that $f(n) = f(n')$ for $n' \leqslant n \leqslant n_i$. using $(++)$ we get $d(z_i) = m \neq \varphi_{z_i}(z_i)$.

Thus we have $d \in DNR_k$ and, therefore, $dg_T(A) \in$ PA. This completes the proof of part (2).

(3) Suppose we are given an index i of the move list of Anke. Let s_1 be the final position of the marker of Anke. Then $s_1 = \lim_{n \to \infty} \varphi_i(n)$. Using a K-oracle we can compute s_1 from i. By hypothesis, there exists $t_1 \in V_2$ adjacent to t_0 with $(s_1, t_1) \in W$. So the list of counter moves $(t_1, t_1, ...)$ wins for Boris.

Now suppose that Boris can A-recursively compute from every index i of a move list of Anke an A-recursive function f_i which is a winning list of counter moves. Let $(v_1, ..., v_k)$ be a winning list of moves for Anke in the off-line version of the finite game.

For any $x_1, ..., x_k$ we define a recursive function $g = g_{x_1, ..., x_k}$ as follows: $g(0) = s_0$, and $g(n) = v_m$, where $m = |\{i : x_i \in K_n\}|$ for $n > 0$. (K_n is the finite set of elements enumerated into K after n steps.)

Now we can A-recursively enumerate for all $x_1, ..., x_k$ a set of at most k strings $\sigma \in \{0, 1\}^k$ such that $F_k^K(x_1, ..., x_k) = (\chi_K(x_1), ..., \chi_K(x_n))$ is among them. By the nonspeedup theorem [1, Theorem 9] (cf. also the equivalent statement in [6, Proposition 4.6]), it follows that $K \leqslant_T A$.

The enumeration procedure works as follows:

Compute an index i of $g = g_{x_1, ..., x_k}$. In step n enumerate $(\chi_{K_n}(x_1,), ..., \chi_{K_n}(x_k))$ if $(g(n), f_i(n)) \in W$. Since f_i wins against g it follows that $(g(n), f_i(n)) \in W$ for all sufficiently large n, so $F_k^K(x_1, ..., x_k)$ is enumerated. Suppose for a contradiction that we enumerate $k+1$ different strings. Choose n_j minimal such that a string with exactly j ones is enumerated in step n_j, $j = 0, ..., k$. Note that $(g(n_0), ..., g(n_k)) = (s_0, v_1, ..., v_k)$.

Then the list of countermoves $(f_i(n_0), ..., f_i(n_k))$ wins against $l = (v_1, ..., v_k)$ in the off-line version of the finite game. This contradicts the hypothesis that l is a winning list of moves. ∎

5. ON THE STRENGTH OF NONINCLUSIONS IN PARALLEL LEARNING

Suppose that (m, n) PFIN $\nsubseteq (h, k)$ PFIN, i.e., there exists a set $S \subseteq$ REC which can be inferred by an (m, n) PFIN-machine, but not by any (h, k) PFIN-machine. What happens if we scale-up the (h, k) PFIN-machines and allow them to access an oracle A? Then S might become (h, k) PFIN-inferable if A is sufficiently powerful. How powerful must A be? Similar questions for inference notions with unbounded mindchanges were studied in [6, 15] and for teams of finite learners in [13]. Let strength$(m, n; h, k)$ denote the class of all such A's. The strength of the noninclusion (m, n) PFIN $\nsubseteq (h, k)$ PFIN is measured by the class strength$(m, n; h, k)$; the stronger the noninclusion the smaller the class strength$(m, n; h, k)$. In our next result we apply Theorem 4.2 and show that there are exactly four possibilities for strength$(m, n; h, k)$. The first one is degenerate, it corresponds to the inclusions (m, n) PFIN $\subseteq$ (h, k) PFIN. The other three cases distinguish between noninclusions. The second case is the noninclusions which can be overcome by a PA-oracle. The third case is the noninclusions which can be overcome, but only with an oracle that decides the halting problem. The fourth case is the noninclusions which cannot be overcome by any oracle.

THEOREM 5.1. *Let* $1 \leqslant m \leqslant n$, $1 \leqslant h \leqslant k$:

(1) strength$(m, n; h, k) = \{A: A \subseteq \omega\}$ *iff* $[n \geqslant k \wedge n - m \leqslant k - h] \vee [n \leqslant k$ *and Boris has a winning strategy in* $G(m, n; h, k)$.

(2) strength$(m, n; h, k) = \{A: dg_T(A) \in \mathrm{PA}\}$ *iff* $[n \leqslant k \wedge$ *Anke has a winning strategy in* $G(m, n; h, k)$, *but Boris has a winning strategy in the off-line version of* $G(m, n; h, k)]$.

(3) strength$(m, n; h, k) = \{A: A \geqslant_T K\}$ *iff* $[n \leqslant k \wedge n - m \leqslant k - h \wedge$ *Anke has a winning strategy in the off-line version of* $G(m, n; h, k)]$.

(4) strength$(m, n; h, k) = \varnothing$ *iff* $n - m > k - h$.

Proof. Since the right-hand sides of (1)–(4) are complete case distinctions, it suffices to show the if-direction in (1)–(4).

(1) If the condition on the right-hand side holds then we have (m, n) PFIN $\subseteq (h, k)$ PFIN by Corollary 1.4 and Theorem 2.3, respectively.

(2) Assume that $n \leqslant k$ and that Boris has a winning strategy for the off-line version of $G(m, n; h, k)$. We show that strength$(m, n; h, k) \supseteq \{A: dg_T(A) \in \mathrm{PA}\}$.

Fix any A with $dg_T(A) \in \mathrm{PA}$. By Theorem 4.2, Boris has a winning strategy for the infinite off-line version of $G(m, n; h, k)$. Similarly as in the proof of Theorem 2.3 ($\Leftarrow$), we can build, for any given (m, n) PFIN-machine M that infers a set $S \subseteq \mathrm{REC}$, an A-recursive (h, k) PFIN-machine N^A which simulates M:

On input $f_1, ..., f_k$ we simulate $M(f_{i_1}, ..., f_{i_k})$ for every n-element subset $D = \{i_1 < \cdots < i_n\} \subseteq \{1, ..., k\}$ until it outputs programs $(e_{D, i_1}, ..., e_{D, i_n})$, for every such D. These programs determine in a uniform way an off-line strategy for Anke in $G(m, n; h, k)$. We compute an index i of this strategy. Now we are using the oracle A to compute a finite list l of counter moves for Boris such that l does not lose against i. This is done as in the proof of Theorem 4.2(2). Only at this point the machine N^A outputs programs for k functions $g_1, ..., g_k$. These are equipped with the move list l which they use in the same way as the winning strategy for Boris was used in the proof of Theorem 2.3 ($\Leftarrow$). By an analogous argument as in this proof it follows that at least h of the g_i's are correct, and all of them are total. $S \in (h, k)$ PFIN$[A]$ via N^A.

Now assume that $n \leqslant k$ and that Anke has a winning strategy in $G(m, n; h, k)$. Fix any oracle A with $dg_T(A) \notin \mathrm{PA}$. We show that $A \notin$ strength$(m, n; h, k)$.

By a modification of the proof of Theorem 2.3 ($\Rightarrow$), we can construct a set $S \in (m, n)$ PFIN $-$ (h, k) PFIN$[A]$. In this proof the basic building block was a uniform strategy of how to diagonalize (h, k)-machines. The uniformity was possible, since we could effectively simulate the (h, k)-machines. Now, when the (h, k)-machines are equipped with a nonrecursive oracle A, an effective simulation is impossible. Instead we define for each (h, k)-machine M^A (but independently of the actions of M^A) an infinite sequence of k-tuples $\{f_1^p, ..., f_k^p\}$ for all $p \geqslant 0$ such that there is a fixed (m, n)-algorithm which infers each k-tuple. We then argue that if M (h, k)-infers each k-tuple, then $A \in \mathrm{PA}$. Thus there is a k-tuple which is not inferred by M. This is chosen (nonuniformly!) and put into S.

More formally (using the notation of Theorem 2.3 ($\Rightarrow$)), we diagonalize a single (h, k)-machine M_i^A by building a uniformly recursive sequence $F_{\langle i, p\rangle, e, D, j}$ for $p \geqslant 0$: The functions $F_{\langle i, p\rangle, e, D, j}$ are defined according to the moves of Anke which are given by the p th recursive off-line strategy strat$_p$. Here we refer to the corresponding listing $\{\mathrm{strat}_p\}_{p \in \omega}$ of recursive off-line strategies for Anke as they are used in the proof of Theorem 4.2(2). Note that, as in the proof of the Theorem 2.3 ($\Rightarrow$), the action of M_i^A defines an A-recursive counter strategy for each strat$_p$. Recall from the proof of Theorem 4.2 (2), that each strat$_p$ wins against some potential strategy of Boris where the moves in each round are correctly predicted by strat$_p$. We have shown there that if one can A-recursively compute for each strat$_p$ a counter strategy which wins against strat$_p$, then $dg_T(A) \in \mathrm{PA}$. The action of M_i^A on the initial segments of the $F_{\langle i, p\rangle, e, D, j}$'s, however, defines an A-recursive counter strategy for Boris.

In order to formally cover the case where the $F_{\langle i, p\rangle, e, D, j}$'s split before M_i^A has produced its guess, we may introduce the convention that the corresponding positions are B-configurations. I.e., if Boris' markers are in node λ and one of Anke's marker is not in $\{1, ..., k\}$ then this is a B-configuration. In particular, Boris wins the off-line version of the infinite game if he keeps his markers in λ and finds a stage where Anke moves. However, if Anke would never move then this strategy would not be successful.

Since $A \notin \mathrm{PA}$ it follows that there exists p such that the strategy provided by M_i^A loses against strat$_p$. This means that we can define $f_1, ..., f_k$ which are not (h, k) PFIN-inferred by M_i^A, but which are inferred in a uniform way by a recursive (m, n) PFIN-algorithm.

As in the proof of Theorem 2.3 ($\Rightarrow$), we define $S \in (m, n)$ PFIN $-$ (h, k) PFIN$[A]$ by pasting together the k-tuples that diagonalize M_i^A for $i \in \omega$.

(3) Assume that $n \leqslant k$ and $n - m \leqslant k - h$. Let an (m, n) PFIN-machine M be given which infers a class $S \subseteq \mathrm{REC}$. We can build a K-recursive (h, k) PFIN-machine N^K which simulates M.

As above, on input $f_1, ..., f_k$ we simulate $M(f_{i_1}, ..., f_{i_k})$ for every n-elements subset $D = \{i_1 < \cdots < i_n\} \subseteq \{1, ..., k\}$ until it outputs programs $(e_{D, i_1}, ..., e_{D, i_n})$, for every such D.

Since the $F_{D,j}$'s are total we can K-recursively compute which of them are equal and which are different. Then we find s_0 such that if two of these functions differ then they differ on an argument less than s_0. If there is a function $F_{D,j}$ which agrees with f_j for all arguments less than s_0 then let $g_j = F_{D,j}$; otherwise let $g_j = \lambda x..0$. We output a k-tuple of programs for $(g_1, ..., g_k)$.

Clearly, every program which we output compute a total function. We claim that at most $n-m$ of them are incorrect. Suppose for a contradiction that there is a set E of $n-m+1$ indices j with $f_j \neq g_j$. Choose an n-element set $D \subseteq \{1, ..., k\}$ with $E \subseteq D$. For every $j \in E$: if $F_{D,j} = g_j$ then $F_{D,j} \neq f_j$, by the hypothesis on g_j; if $F_{D,j} \neq g_j$ then $F_{D,j} \neq f_j$, since $F_{D,j}$ must already differ from f_j on some argument less than s_0. Thus more than $n-m$ of the $F_{D,j}$'s are incorrect; i.e., M does not (m, n)-infer $\{f_i : i \in D\}$, a contradiction. This shows that N^A makes at most $n-m \leqslant k-h$ errors; i.e., it (h, k)-infers S.

Finally, assume that $n \geqslant k$, Anke has a winning strategy in the off-line version of $G(m, n; h, k)$, and (m, n) PFIN $\subseteq$ (h, k) PFIN$[A]$. Then $A \geqslant_T K$. This is shown by combining the proofs of Theorem 2.3 with the proof of Theorem 4.2 in a similar way as in (3) above. We omit the details.

(4) This follows from the observation that the diagonalization in the proof of Fact 1.3 in [11] also works against (h, k) PFIN-algorithms which have access to an oracle. ∎

Each of these four cases occur in a nontrivial way:

(1) (4, 5) PFIN $\subseteq$ (5, 6) PFIN; see Proposition 3.2.

(2) This holds for (2, 3) PFIN versus (3, 4) PFIN$[A]$; one can check that Boris has a winning strategy for the off-line version of the game $G(2, 3; 3, 4)$ (cf. the proof of Proposition 3.1).

(3) This holds if $n \leqslant k$, $n-m \leqslant k-h$, and there is an (m, n)-admissible set which is not (h, k)-admissible; the proof of Theorem 3.9 actually provides an off-line winning strategy for Anke. For example, this holds for PFIN(1, 3) versus PFIN(2, 5)$[A]$.

(4) Obvious.

When we look back at the paper [13] from the game theoretic point of view, it turns out that all noninclusions for popperian teams of finite learners can be shown by off-line strategies (cf. [13, Section 5]). This explains, why a K-oracle is necessary to overcome any of these noninclusions. In contrast, for general teams of finite learners it was shown that to overcome the noninclusion $[24, 49]\ EX_0 \nsubseteq [2, 4]\ EX_0$, PA-oracles are necessary and sufficient. Indeed, the diagonalization strategy of [3] appears to be "intrinsically on-line." This is only an heuristic explanation, since we still do not have a finitary game theoretic characterization of the inclusion problem for teams of finite learners—hence we cannot use Theorem 4.2 directly to prove the 24/49-result in [13]. We conjecture, however, that such a characterization is possible.

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